Boundedness of dynamical systems and chaos synchronization

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Chaos synchronization in bounded dynamical systems is studied. We use boundedness of trajectories within the nonreplica approach to chaos synchronization and the Routh-Hurwitz criterion to propose a simple method to make negative conditional Lyapunov exponents. The method is tested on the classical Lorenz model. [S1063-651X(99)05910-3]

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In recent years, synchronization of chaotic systems has become an area of active research, especially in light of its potential application in secure communications, modeling brain activity, etc. [1–9]. Usually two dynamical systems are called synchronized if the distance between their states converges to zero for $t\rightarrow\infty$. Recently [10,11], a generalization of this concept was proposed, where two systems are called synchronized if a functional relation exists between the states of both systems. Different approaches to chaos synchronization have been proposed. In their seminal paper, Pecora and Carroll [1] show that when a state variable from a chaotically evolving system is transmitted as an input (driving variable) to a replica of part of the original system, the replica subsystem (driven subsystem or receiver) sometimes synchronizes to the original system (driving system or sender). The occurrence of this synchronization is conditioned on whether the subsystem's Lyapunov exponents are negative. Because of their dependence on the driving variable, it has been suggested [2] that they be called conditional Lyapunov exponents. In a related paper [12], it was pointed out that in many representative cases, chaos synchronization can be understood from the existence of a global Lyapunov function of the difference signals.

Recently in [8] it was shown that the receiver subsystem does not need to be a replica of part of the sender system. According to [8], synchronization between the original dynamical system

$$\frac{d\mathbf{x}}{dt}=\mathbf{G}(\mathbf{x},\mathbf{y}),\qquad \frac{d\mathbf{y}}{dt}=\mathbf{H}(\mathbf{x},\mathbf{y}), \tag{1}$$

(where in general $\mathbf{x}$ and $\mathbf{y}$ are vectors with high dimension; usually the dimensionality of the driving variable $\mathbf{x}$ is equal to unity) and the nonreplica response system

$$\frac{d\mathbf{x}_{nr}}{dt}=\mathbf{K}(\mathbf{x},\mathbf{x}_{nr},\mathbf{y}_{nr}),\qquad \frac{d\mathbf{y}_{nr}}{dt}=\mathbf{L}(\mathbf{x},\mathbf{x}_{nr},\mathbf{y}_{nr}), \tag{2}$$

is also possible, provided that functions $\mathbf{K}$ and $\mathbf{L}$ satisfy the conditions

$$\mathbf{K}(\mathbf{x},\mathbf{x},\mathbf{y})=\mathbf{G}(\mathbf{x},\mathbf{y}),\qquad \mathbf{L}(\mathbf{x},\mathbf{x},\mathbf{y})=\mathbf{H}(\mathbf{x},\mathbf{y}), \tag{3}$$

In fact, allowing the use of nonreplica systems adds flexibility and enhances the possibility of synchronization [8]. In this paper we will use the term ''nonreplica approach'' when a nonreplica response system is considered; the case in which the response system is a replica of part of the original system is referred to as the replica approach.

This article deals with chaos synchronization in dynamical systems whose solutions are bounded. We use the boundedness of solutions of dynamical systems, the nonreplica approach to chaos synchronization [8], and the Routh-Hurwitz criterion to make negative all the conditional Lyapunov exponents without complex numerical and analytical calculations. Consider a general dynamical system that displays chaotic behavior:

$$\begin{aligned}\frac{dx_1}{dt}&=f_1(x_1,x_2,\ldots,x_N),\\ \frac{dx_2}{dt}&=f_2(x_1,x_2,\ldots,x_N),\\ &\vdots\\ \frac{dx_N}{dt}&=f_N(x_1,x_2,\ldots,x_N),\end{aligned} \tag{4}$$

where $x_1,x_2,\ldots,x_N$ are state variables, $f_1,f_2,\ldots,f_N$ are sufficiently smooth functions of $x_1,x_2,\ldots,x_N$. Without loss of generality, take the state variable x_1 as a driver. Then using the approach developed in [8] we construct the following nonreplica response system (with the subscript nr):

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$$\frac{dx_{nr1}}{dt}=f_1(x_1,x_{nr2},\dots,x_{nrN})+c_1(x_{nr1}-x_1)=F_1,$$
$$\frac{dx_{nr2}}{dt}=f_2(x_1,x_{nr2},\dots,x_{nrN})+c_2(x_{nr1}-x_1)=F_2,\tag{5}$$
$$\vdots$$
$$\frac{dx_{nrN}}{dt}=f_N(x_1,x_{nr2},\dots,x_{nrN})+c_N(x_{nr1}-x_1)=F_N.$$

Here it is necessary to emphasize that, in order to construct the nonreplica response system, we added to the right-hand side of the original nonlinear equations some damping terms with coupling or damping strength c_i that vanish when chaos synchronization is achieved. In fact, we use linear feedback functions. The selected functions are of the simplest possible form. Of course, one could select other types of functions satisfying conditions (3); however, the simplest linear feedback functions will suffice for our purpose.

The eigenvalues of the Jacobian matrix of the nonreplica system (5),

$$J=\frac{\partial(F_1,F_2,\dots,F_N)}{\partial(x_{nr1},x_{nr2},\dots,x_{nrN})},\tag{6}$$

satisfy the following equation:

$$\lambda^N+a_1\lambda^{N-1}+\cdots+a_{n-1}\lambda+a_N=0,\tag{7}$$

where $a_1,a_2,\dots,a_N$ are, in general, functions of the arbitrary constants c_1 and $x_1(t),x_2(t),\dots,x_N(t)$ are the solutions of the original nonlinear system (4). Our task is to make negative all the roots of Eq. (7) without the need to perform complex numerical and analytical calculations. It appears that the reasonably large class of dynamical systems with bounded solutions is suitable for our purposes. As shown by Lorenz in [13], dissipative systems of the form

$$\frac{dx_i}{dt}=\sum_{j,k=1}^{N}a_{ijk}x_jx_k-\sum_{j=1}^{N}b_{ij}x_j+d_i,\tag{8}$$

with the constants chosen so that $\Sigma a_{ijk}x_ix_jx_k$ vanishes identically and $\Sigma b_{ij}x_ix_j$ is positive definite, have bounded solutions.

The problem of determining the roots of Eq. (7) when $N>2$ can become tedious at best. Fortunately, what is required is not these roots, but simply the region of c_i in which all the roots of Eq. (7) become negative. The answer to this problem is well known. A necessary and sufficient condition for all roots of the polynomial equation $f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n=0$ to have negative real parts is that all Hurwitz determinants $\Delta_1,\Delta_2,\dots,\Delta_n$ are positive [14]. In the case of third-order equation $(n=3)$, these determinants are

$$\Delta_1=a_1,\quad \Delta_2=\begin{vmatrix}a_1 & a_3\\ a_0 & a_2\end{vmatrix},\quad \Delta_3=\begin{vmatrix}a_1 & a_3 & a_5\\ a_0 & a_2 & a_4\\ 0 & a_1 & a_3\end{vmatrix},\tag{9}$$

(with $a_4=a_5=0$; in addition, $a_0=1$ in our case.). Note that the positiveness of the $\Delta_1=a_1$, $\Delta_2=a_1a_2-a_3$, $\Delta_3=a_3\Delta_2$ implies that the Routh-Hurwitz criterion can be written as $a_1>0,a_3>0,a_1a_2-a_3>0$. We use the boundedness of the solutions of the dynamical system and combine it with the Routh-Hurwitz criterion to control the sign of real parts of the roots of Eq. (7) without conducting explicit calculations. Our argument is based on an analysis of the system consisting of the variable differences between the original and response dynamical systems, which we refer to as ''error dynamics.'' In fact the essence of our approach is simple: just choose to oblige the variable of the response system, which is the same as the drive variable, to synchronize, i.e., to relax towards the appropriate value by damping the difference very strongly. We give a concrete example to explain the process in detail.

Consider the Lorenz system [15–18]:

$$\frac{dx}{dt}=\sigma(y-x),$$
$$\frac{dy}{dt}=rx-y-xz,\tag{10}$$
$$\frac{dz}{dt}=xy-bz.$$

The parameters used for chaotic behavior by Lorenz and most other investigators are $\sigma=10$ and $b=\frac{8}{3}$; r must be larger than a critical Rayleigh number of r_{cr}. For $\sigma=10$, $b=\frac{8}{3}$, $r_{\mathrm{cr}}=24.74$. Throughout this paper, we take $r=60$.

We will consider dynamical variables x, y, and z as the drivers and in all cases we rewrite the Lorenz system so that it begins with the equation for the driving variable, which will be denoted x_1. We begin with the case $x_1\equiv x$ and put $x_2\equiv y$, $x_3\equiv z$. Consider the following nonreplica response system:

$$\frac{dx_{nr1}}{dt}=\sigma(x_{nr2}-x_1)+c_1(x_{nr1}-x_1),$$
$$\frac{dx_{nr2}}{dt}=rx_1-x_{nr2}-x_1x_{nr3}+c_2(x_{nr1}-x_1),\tag{11}$$
$$\frac{dx_{nr3}}{dt}=x_1x_{nr2}-bx_{nr3}+c_3(x_{nr1}-x_1).$$

As the calculations show, the eigenvalues of the Jacobian matrix of the system (11) satisfy the following equation:

$$\lambda^2+\lambda^2(b+1-c_1)+\lambda[b-c_1(b+1)+x_1^2-c_2b]+\sigma c_3x_1-x_1^2c_1-c_2\sigma b-c_1b=0.\tag{12}$$

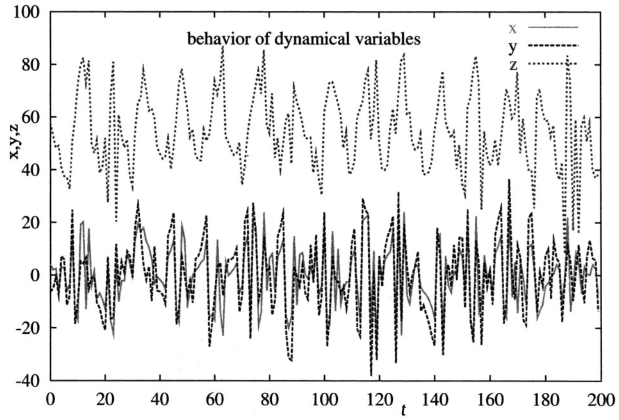

FIG. 1. Temporal evolution of dynamical variables in the Lorenz model for $\sigma=10$, $b=\frac{8}{3}$, $r=60$. (All ordinate values are dimensionless.)

Here $x_1(t)$, $x_2(t)$, and $x_3(t)$ are the solutions of the Lorenz system (10) with $x_1\equiv x$, $x_2\equiv y$, $x_3\equiv z$. For simplicity, we take $c_2=c_3=0$. Then from Eq. (12) it is easy to establish that for negative c_1 conditional Lyapunov exponents become negative. Indeed, for negative c_1 the Routh-Hurwitz criterion $a_1=b+1-c_1>0$, $a_3=-c_1(b+x_1^2)>0$, $a_1a_2-a_3=(b-c_1)^2-c_1b(b-c_1)+x_1^2b+b-c_1+x_1^2>0$ is satisfied. The global asymptotic stability can be investigated using the Lyapunov function approach [12]. For error dynamics $e_1=x_{nr1}-x_1$, $e_2=x_{nr2}-x_2$, $e_3=x_{nr3}-x_3$,

$$\frac{de_1}{dt}=\sigma e_2+c_1e_1,$$

$$\frac{de_1}{dt}=-e_2-x_1e_3+c_2e_1, \tag{13}$$

$$\frac{de_3}{dt}=x_1e_2-be_3+c_3e_1,$$

one can use the Lyapunov function

$$E=\tfrac{1}{2}(e_1^2+e_2^2+e_3^2). \tag{14}$$

Since

$$\frac{dE}{dt}=-e_2^2-be_3^2+\sigma e_1e_2+c_1e_1^2+c_2e_1e_2+c_3e_1e_3 \tag{15}$$

can be made strictly negative for $c_3=0$, $c_2=-\sigma$, and negative c_1, we conclude that the asymptotic stability is global.

Now consider the y variable in the original Lorenz model as a driver. To maintain consistent notation throughout this paper, we reorder variables and equations: $x_1\equiv y$, $x_2\equiv x$, $x_3\equiv z$, which yields for the nonreplica system

$$\frac{dx_{nr1}}{dt}=rx_{nr2}-x_1-x_{nr2}x_{nr3}+c_1(x_{nr1}-x_1),$$

$$\frac{dx_{nr2}}{dt}=\sigma(x_1-x_{nr2})+c_2(x_{nr1}-x_1), \tag{16}$$

$$\frac{dx_{nr3}}{dt}=x_1x_{nr2}-bx_{nr3}+c_3(x_{nr1}-x_1).$$

The eigenvalues of the Jacobian matrix satisfy the following equation:

$$\lambda^3+\lambda^2(\sigma+b-c_1)+\lambda[x_2c_3-c_2(r-x_3)+b\sigma-c_1(b+\sigma)] -c_1b\sigma+x_1x_2c_2+c_3x_2\sigma-c_2b(r-x_3)=0, \tag{17}$$

An inspection of Eq. (17) suggests that the Routh-Hurwitz criterion is implementable if c_1 is negative. In the simplest case, we put $c_2=0$, $c_3=0$ and obtain that $\lambda_1=-\sigma$, $\lambda_2=-b$, $\lambda_3=c_1$. Thus, for negative c_1 all conditional Lyapunov exponents are negative and synchronization between driving and response systems takes place. Analyzing the error dynamics, one can also show that synchronization occurs for all initial conditions. The equations for the error dynamics are

$$\frac{de_1}{dt}=re_2-e_2e_3-x_3e_2-x_2e_3+c_1e_1,$$

$$\frac{de_2}{dt}=-\sigma e_2+c_2e_1, \tag{18}$$

$$\frac{de_3}{dt}=x_1e_2-be_3+c_3e_1.$$

Here $x_1(t)$, $x_2(t)$, and $x_3(t)$ are the solutions of the Lorenz system (10) with $x_1\equiv y$, $x_2\equiv x$, $x_3\equiv z$. Now we use the fact that solutions to the Lorenz system are bounded. The bounding value depends on the relationships between the system's parameters, and its expression can be found in different textbooks and papers; see, e.g., [15–17,19]. For example, according to [19], the solutions to the Lorenz model always satisfy inequality $x^2(t)+y^2(t)+[z(t)-r-\sigma]^2\leqslant(\sigma+r)^2K^2$, where $K^2=\frac{1}{4}+(b/4)\max(\sigma^{-1},1)$. Estimations show that for the above-mentioned values of parameters, the maximal value of the dynamical variables is theoretically of the order of $2(r+\sigma)$; so for $r=60$, $\sigma=10$ we can put safely $x(t)<140$, $y(t)<140$, $z(t)<140$. (See Fig. 1.) Since $x_{1,2,3}$ are bounded and one can choose the magnitude and sign of the arbitrary constants in Eq. (18), it can easily be seen that for "sufficiently negative" c_1, the differences e_1, e_2, and e_3 approach zero. At extremely large negative c_1 we can slave x_{nr1} to x_1. This is like replacing all occurrences of x_{nr1} in the response with x_1. Thus, for $c_1\to-\infty$ we asymptotically approach the replica method of synchronization; we obtain that in the limiting case of $c_1\to-\infty$ the equations for e_2,e_3 describe error dynamics within the replica approach in the case of x_1 driving. And as noted in [4], the replica response system (x_2,x_3) is globally asymptotically stable. Usually, global asymptotic stability is studied using the Lyapunov function approach [12]. However, in some cases the analysis of error dynamics is the simplest and most

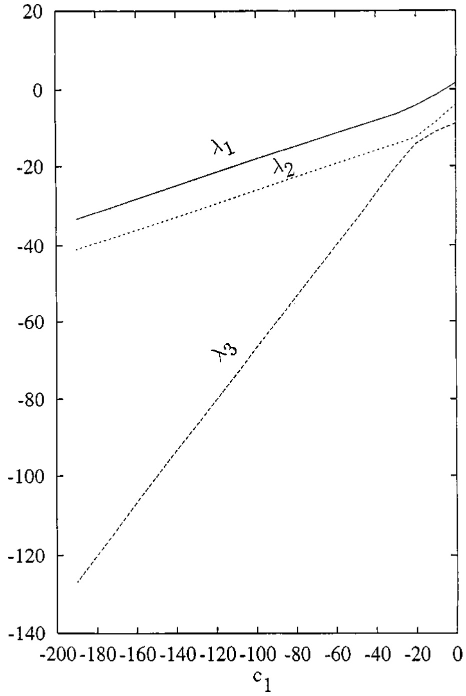

FIG. 2. Conditional Lyapunov exponents as a function of c_1 ($c_2=c_3=0$). (All ordinate values are dimensionless.)

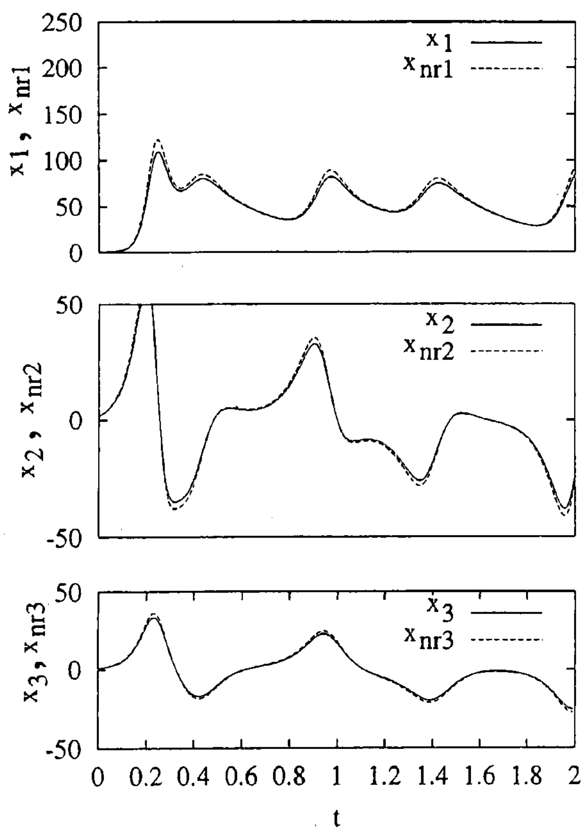

FIG. 3. Temporal evolution of dynamical variables described by Eqs. (21) and (22) for $c_1=-10$, $c_2=c_3=0$ with $x_{nr1}(0)-x_1(0)=x_{nr2}(0)-x_2(0)=x_{nr3}(0)-x_3(0)=0.1$. (All ordinate values are dimensionless.)

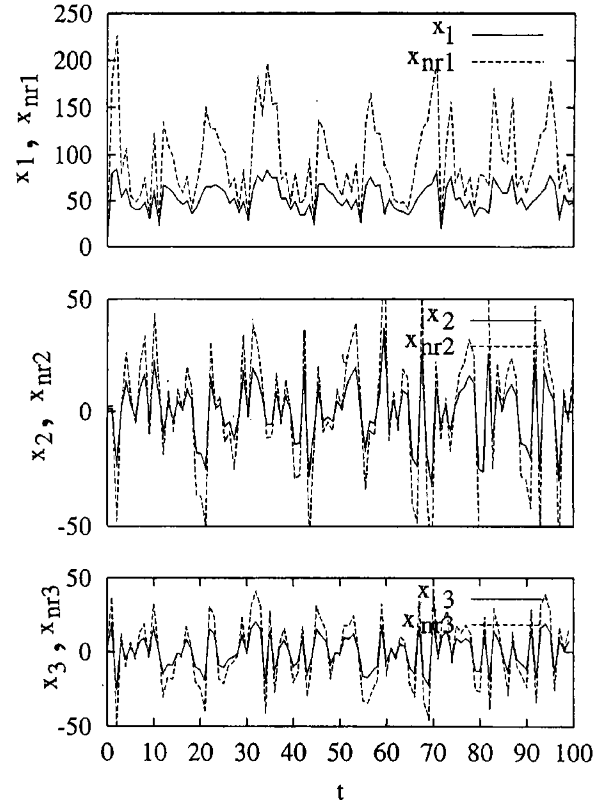

FIG. 4. Same as in Fig. 3, but with $x_{nr1}(0)-x_1(0)=x_{nr2}(0)-x_2(0)=x_{nr3}(0)-x_3(0)=1$. (All ordinate values are dimensionless.)

straightforward way of establishing global asymptotic stability. Indeed, with $x_1 \equiv y$ driving within the replica approach, the error dynamics is described by equations

$$\frac{de_2}{dt}=-\sigma e_2,$$
$$\frac{de_3}{dt}=x_1e_2-be_3. \tag{19}$$

Using the Lyapunov function of the form $E=\frac{1}{2}(e_2^2+e_3^2)$, we obtain that $dE/dt=-\sigma e_2^2-be_3^2+x_1e_2e_3$, from which it is very difficult to judge (at least analytically) how to make dE/dt strictly negative in the general case. But from error dynamics itself (19) it is immediately obvious that e_2 and e_3 approach zero.

Applying this approach, we establish that for sufficiently negative c_1, e_1 approaches zero. Inserting $e_1=0$ in the last equation in (18), one can see that e_2 and e_3 also approach zero.

Finally, consider the case of z driving in the original Lorenz system. Reordering variables and equations $x_1\equiv z$, $x_2 \equiv y$, $x_3\equiv x$ in the original model (10), we get the following nonreplica system:

$$\frac{dx_{nr1}}{dt}=x_{nr2}x_{nr3}-bx_1+c_1(x_{nr1}-x_1),$$
$$\frac{dx_{nr2}}{dt}=rx_{nr3}-x_{nr2}-x_{nr3}x_1+c_2(x_{nr1}-x_1), \tag{20}$$
$$\frac{dx_{nr3}}{dt}=\sigma(x_{nr2}-x_{nr3})+c_3(x_{nr1}-x_1).$$

Conditional Lyapunov exponents are to be found from the equation

$$\lambda^3+\lambda^2(\sigma+1-c_1)-\lambda[x_2c_3+x_3c_2+(\sigma+1)c_1 +\sigma(r-x_1)-\sigma]-x_2(\sigma c_2+c_3)-\sigma c_1-x_3c_2\sigma -(r-x_1)(x_3c_3-c_1\sigma)=0, \tag{21}$$

with $x_1(t)$, $x_2(t)$, and $x_3(t)$ being the solutions of the original Lorenz model (10): $x_1\equiv z$, $x_2\equiv y$, $x_3\equiv x$. In this case, we failed to make negative conditional Lyapunov exponents analytically by choosing arbitrary constants c_i as easily as in previous cases; so we used numerical simulations (Fig. 2).

We obtain the following error dynamics:

$$\frac{de_1}{dt}=e_2e_3+x_3e_2+x_2e_3+c_1e_1,$$

$$\frac{de_2}{dt}=re_3-e_2-x_1e_3+c_2e_1, \tag{22}$$

$$\frac{de_3}{dt}=\sigma(e_2-e_3)+c_3e_1.$$

Since we also failed to demonstrate analytically the global asymptotic stability using the Lyapunov function approach, here we present the results of numerical simulations and a direct analysis of the error dynamics itself. From the first equation of (22) one can establish that for sufficiently negative c_1 the difference e_1 approaches zero, since the solutions of the Lorenz system are bounded and the choice of constants c_i is arbitrary. At extremely large negative c_1, the error e_1 cancels, and we arrive at the generalized synchronization case [20]; see also [21]. As reported in [20], the subsystem (x_2,x_3) is uniformly stable.

Our numerical results show that only for equal or very close initial conditions (when differences between initial conditions do not exceed 0.001) can we obtain perfect convergence between the driving and response systems. For arbitrary initial conditions, we established that trajectories of driving and response systems do not diverge in time (Figs. 3 and 4). From these figures one can see that the temporal evolution of the response system repeats that of the driving system exactly (but with different amplitudes). We conclude that in this case, the type of generalized synchronization reported in [20,21] for the limiting case of $c_1\rightarrow-\infty$ also occurs for moderately negative values of c_1. With increasingly negative c_1, $e_1=0$ is obtained with an increasingly higher degree of accuracy, thus making decoupling of the subsystem (e_2,e_3) from Eq. (22) more conspicuous. In other words, we thus conjecture that the system (nonreplica response system) retains the memory of its part (replica response system). And this conjecture is valid not only for the case of z driving, but also for the cases of x and y driving, since we have shown that in the latter case both replica and nonreplica response systems are globally asymptotically stable.

We also successfully applied the present approach in cases where the nonreplica response system has a constant Jacobian, such as in the Rössler system [22,23] (driving by x) and the four-dimensional Duffing oscillator [24] (driving by x). These systems contain only one nonlinear term, which is a function of a single variable. Considering this variable as a driver, we obtain a response system with a constant Jacobian.

In conclusion, we use the nonreplica approach to chaos synchronization and boundedness of trajectories of the dynamical system, and combine it with the Routh-Hurwitz criterion to control the sign of real parts of conditional Lyapunov exponents.

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